AIAA 78-1441R

Geosynchronous Satellite Perturbations Due to Earth's Triaxiality and Luni-Solar Effects

Ahmed Aly Kamel*
Ford Aerospace and Communication Corporation, Palo Alto, Calif.

Geosynchronous satellite perturbations are developed in terms of equinoctial orbital elements. The Earth's triaxiality effect is represented by zonal and tesseral harmonics up to J_{33} coefficients. Lunar motion is represented by the Hill-Brown theory with coefficients up to 10^{-3} rad. The solar motion is described by an elliptic orbit expansion with terms up to 10^{-4} rad. The equations of motion are described by a Hamiltonian system that leads to three decoupled sets of simple-pendulum equations with forced oscillations. The basic periods of these forced oscillations are essentially the satellite-diurnal, lunar-monthly, solar-yearly, lunar perigee 8.9-year progression and lunar node 18.6-year regression.

Nomenclature

a	= orbital semimajor axis
a_m	= mean distance of the moon = 3.844E05 km
a_s	= mean distance of the sun = 1.496E08 km
a_{syn}	= synchronous semimajor axis = $(\mu_e/\omega_e^2)^{1/3}$ = 42164.176 km
e	= orbital eccentricity
e_1	= $e\cos(\omega+\Omega)$
e_2	= $e\sin(\omega+\Omega)$
g_1	= $l-\bar{l}$
g_2	= $\frac{1}{2}(a/a_{syn}-1)$
GHA	= Greenwich hour angle = Greenwich right ascension measured from the mean equinox of epoch = $\omega_e t + \mathrm{GHA}_0$
GHA_0	= Greenwich hour angle at epoch
H	= perturbing Hamiltonian
h_1	= $\sin i\cos\Omega$
h_2	= $\sin i\sin\Omega$
i	= orbital inclination
J_{ij}	= ij coefficient in Earth's potential function: numerical values are taken as [16,17] $J_{20}=1.082637\mathrm{E}-03$, $J_{22}=1.771156\mathrm{E}-06$, $J_{31}=2.171210\mathrm{E}-06$, $J_{33}=2.238015\mathrm{E}-07$
l	= satellite mean longitude measured from the mean equinox of epoch along the equator to the ascending node of the satellite orbit, then along the orbit = $\Omega+\omega+M$
$\bar{l}$	= $\mathrm{GHA}+\lambda_{syn}$
M	= orbital mean anomaly
R_e	= Earth's equatorial radius = 6378.140 km
t	= time in days from epoch
λ_{ij}	= ij angle in Earth's potential function: numerical values, in radians, are taken as [16,17] $\lambda_{22}=-0.2604681$, $\lambda_{31}=0.1113511$, $\lambda_{33}=0.4201560$
λ_{syn}	= station right ascension measured from Greenwich along the equator (station longitude)
μ_e	= Earth's gravitational constant = 398600.64 $\mathrm{km}^3/\mathrm{s}^2$
μ_m	= moon's gravitational constant = $\mu_e/81.3$ = 4902.837 $\mathrm{km}^3/\mathrm{s}^2$
μ_s	= sun's gravitational constant = 328912.0 $(\mu_e+\mu_m)$ = 1.327179E11 $\mathrm{km}^3/\mathrm{s}^2$
ω	= argument of perigee
ω_e	= Earth's sidereal angular velocity = 6.300388 rad/day
Ω	= right ascension of ascending node

Presented as Paper 78-1441 at the AIAA/AAS Astrodynamics Conference, Palo Alto, Calif., Aug. 7-9, 1978; submitted April 22, 1980; revision received Aug. 14, 1981.

*Supervisor of Orbit Dynamics Section, Mission Analysis Department. Member AIAA.

I. Introduction and Overview

In his recent exhaustive survey on the orbital perturbations of geosynchronous satellites, Shrivastava[1] felt the need for a more complete investigation that combines the effects of various perturbations on geosynchronous satellites. In the present paper, this goal has been achieved by using the REDUCE[2,3] computer program to perform the enormous algebraic manipulations needed in the course of the required analysis. Considerable simplification of the equations of motion has been made due to the fact that geosynchronous satellites are kept near a prescribed orbit by means of active control (stationkeeping maneuvers). In Sec. II, these simplified equations are developed in terms of a Hamiltonian system described by equinoctial elements similar to those used by Cefola and Broucke.[4,5] The Earth's triaxiality part of the perturbation is obtained from Refs. 6-8, while the luni-solar effects are similar to those used in Refs. 9-13. The sun and moon ephemerides were developed in an earlier paper[14] in a form suitable for substitution in the luni-solar perturbing function. The satellite equations of motion are obtained by substituting the satellite position and the sun-moon ephemerides according to the ordering procedure described in Sec. III. The solution of the equations of motion is developed in Secs. IV and V, and it is shown to be determined by three decoupled sets of simple-pendulum-type equations with forced oscillations. The largest forced amplitudes are essentially those with 8.9- and 18.6-yr periods. The eccentricity vector has an amplitude of about 2×10^{-4} with the 8.9-yr lunar perigee progression period. Both the angular momentum vector and the satellite longitude have largest amplitudes of about 0.5 deg with the 18.6-yr lunar node regression period.

For satellites with large-area-to-mass ratio, the solar radiation pressure effect on the satellite motion becomes significant. This effect has been investigated in a separate paper[15] whose final results can be easily superimposed on the solution given in the present paper.

II. Variation of Orbital Elements

For near-synchronous satellite orbits, Lagrange's equations can be approximated by the following Hamiltonian system:

$$\dot{h}_1=-\partial H/\partial h_2 \qquad \dot{h}_2=\partial H/\partial h_1$$
$$\dot{e}_1=-\partial H/\partial e_2 \qquad \dot{e}_2=\partial H/\partial e_1 \tag{1}$$
$$\dot{g}_1=-\partial H/\partial g_2 \qquad \dot{g}_2=\partial H/\partial g_1$$

The significant parts of the perturbing Hamiltonian can be put in the form

$$H=(3/2)\omega_e g_2^2+R_{\text{obl}}+R_{\text{tess}}+R_m+R_s \tag{2}$$

where the R's are the disturbing functions given by

$$R_{\text{obl}}=\epsilon_{\text{obl}}(a/r)^3[1/3-2g_2-z^2] \tag{3}$$

$$R_{\text{tess}}=-\epsilon_{\text{tess}}[G_1g_1+\tfrac{1}{2}G_2g_1^2+G_3g_2] \tag{4}$$

$$G_1=4\sin 2(\lambda_{\text{syn}}-\lambda_{22})-\frac{J_{31}}{J_{22}}\left(\frac{R_e}{a_{\text{syn}}}\right)\sin(\lambda_{\text{syn}}-\lambda_{31})+30\frac{J_{33}}{J_{22}}\left(\frac{R_e}{a_{\text{syn}}}\right)\sin 3(\lambda_{\text{syn}}-\lambda_{33})$$

$$G_2=8\cos 2(\lambda_{\text{syn}}-\lambda_{22})-\frac{J_{31}}{J_{22}}\left(\frac{R_e}{a_{\text{syn}}}\right)\cos(\lambda_{\text{syn}}-\lambda_{31})+90\frac{J_{33}}{J_{22}}\left(\frac{R_e}{a_{\text{syn}}}\right)\cos 3(\lambda_{\text{syn}}-\lambda_{33})$$

$$G_3=12\cos 2(\lambda_{\text{syn}}-\lambda_{22})-8\frac{J_{31}}{J_{22}}\left(\frac{R_e}{a_{\text{syn}}}\right)\cos(\lambda_{\text{syn}}-\lambda_{31})+80\frac{J_{33}}{J_{22}}\left(\frac{R_e}{a_{\text{syn}}}\right)\cos 3(\lambda_{\text{syn}}-\lambda_{33})$$

$$R_m=\epsilon_m\left\{R_{m2}+\left(\frac{a_{\text{syn}}}{a_m}\right)R_{m3}+\left(\frac{a_{\text{syn}}}{a_m}\right)^2R_{m4}+2g_2\left[2R_{m2}+3\left(\frac{a_{\text{syn}}}{a_m}\right)R_{m3}+4\left(\frac{a_{\text{syn}}}{a_m}\right)^2R_{m4}\right]\right\} \tag{5}$$

$$R_s=\epsilon_s[R_{s2}+4g_2R_{s2}] \tag{6}$$

The $R_{mi}(i=2,3,4)$ and R_{s2} are given, in terms of Legendre functions, in the form

$$R_{m2}=\frac{1}{2}\left(\frac{r}{a}\right)^2\left(\frac{a_m}{r_m}\right)^3[3C_m^2-1]$$

$$R_{m3}=\frac{1}{2}\left(\frac{r}{a}\right)^3\left(\frac{a_m}{r_m}\right)^4[5C_m^3-3C_m]$$

$$R_{m4}=\frac{1}{8}\left(\frac{r}{a}\right)^4\left(\frac{a_m}{r_m}\right)^5[35C_m^4-30C_m^2+3] \tag{7}$$

$$R_{s2}=\frac{1}{2}\left(\frac{r}{a}\right)^2\left(\frac{a_s}{r_s}\right)^3[3C_s^2-1]$$

$$C_m=x_mx+y_my+z_mz \qquad C_s=x_sx+y_sy+z_sz$$

where (x,y,z) are the components of the unit vector pointing from the center of the Earth to the satellite; (x_m, y_m, z_m) are the components of the unit vector pointing from the center of the Earth to the moon; and (x_s,y_s,z_s) are the components of the unit vector pointing from the center of the Earth to the sun.

The above three unit vectors will be expressed in the geocentric coordinate system with the x axis along the mean equinox of epoch and z axis along the polar axis of the Earth.

The ratio of the satellite radius to the orbital semimajor axis (r/a) is given, in terms of the true longitude l_1 and (e_1,e_2) by

$$r/a=(1-e_1^2-e_2^2)(1+e_1\cos l_1+e_2\sin l_1)^{-1} \tag{8}$$

where $l_1=\Omega+\omega+f$, and f is the true anomaly.

The small parameters ϵ's of Eqs. (3-6) are defined as follows:

$$\epsilon_{\text{obl}}=(3/2)\omega_eJ_{20}(R_e/a_{\text{syn}})^2=2.341226\text{E}-04\text{ rad/day}$$

$$\epsilon_{\text{tess}}=(3/2)\omega_eJ_{22}(R_e/a_{\text{syn}})^2=3.830163\text{E}-07\text{ rad/day}$$

$$\epsilon_m=\mu_m/a_m^3\omega_e=1.022728\text{E}-04\text{ rad/day}$$

$$\epsilon_s=\mu_s/a_s^3\omega_e=4.696723\text{E}-05\text{ rad/day}$$

Now, to obtain a perturbation solution of Lagrange's Eq. (1), it is essential to express (x_m, y_m, z_m), a_m/r_m, (x_s,y_s,z_s), and a_s/r_s as function of time. This is done in Appendix A of Ref. 14. It is also necessary to express (x,y,z) and a/r as function of the orbital elements defined by Eq. (1). The components (x,y,z) can be expressed, in terms of (h_1,h_2) and the true longitude l_1, as

$$\begin{aligned}x&=\cos l_1+[h_1h_2\sin l_1-h_2^2\cos l_1][1+\sqrt{1-h_1^2-h_2^2}]^{-1}\\ y&=\sin l_1-[h_1^2\sin l_1-h_1h_2\cos l_1][1+\sqrt{1-h_1^2-h_2^2}]^{-1}\\ z&=h_1\sin l_1-h_2\cos l_1\end{aligned} \tag{9}$$

To express Eqs. (8) and (9) in terms of the mean longitude l, substitution of l_1 in terms of l and (e_1,e_2) is needed. This can be achieved either by iteration procedure that solves Kepler time equation or by means of approximate power series of l_1 in terms of (l,e_1,e_2), using the equation of the center as shown in the next section.[18]

III. Expansion of the Disturbing Functions

Significant terms in a disturbing function have to be selected through an ordering procedure. de Pontécoulant found that a small parameter $m\cong 0.07$ (ratio of the sun to the moon mean motions) is convenient for the development of lunar theory.[19] Since the lunar perturbations are used in this paper, it is essential to use a similar parameter to select the significant terms in the perturbation expansion. In the present paper, however, the sun and moon ephemerides are represented by trigonometric series with numerical coefficients as shown in Appendix A of Ref. 14. In this case, the ordering process could be simplified by selecting the small parameter as a rounded decimal number. This suggests that the de Pontécoulant small parameter be rounded to 0.1. The following ordering procedure can, now, be introduced: A number N is said to be of $O(m^i)$ if it satisfies the following inequality:

$$(0.1)^{i+1}<N\le(0.1)^i \tag{10}$$

The ordering sequence of Eq. (10) can be described by the following logarithmic scale:

$O(m^5)$	$O(m^4)$	$O(m^3)$	$O(m^2)$	$O(m)$	$O(1)$	$O(m^{-1})$
10^{-6} – 10^{-5}	10^{-5} – 10^{-4}	10^{-4} – 10^{-3}	10^{-3} – 10^{-2}	10^{-2} – 10^{-1}	10^{-1} – 1	1 – 10

An important property of the decimal ordering system is the following:

$$(0.1)^{i+j+2}<O(m^i)O(m^j)\le(0.1)^{i+j} \tag{11}$$

According to the above ordering sequence, the sun ephemeris of Appendix A of Ref. 14 is given up to $O(m^3)$ (because all coefficients are $>10^{-4}$) and the moon ephemeris is given up to $O(m^2)$. The small parameters ϵ's are of the following orders: ϵ_{obl} and $\epsilon_m = O(m^3)$, $\epsilon_s = O(m^4)$, $\epsilon_{tess} = O(m^6)$. The parallax ratio $a_{syn}/a_m = 1.096885E-01 = O(1)$ and $(a_{syn}/a_m)^2 = 1.203157E-02 = O(m)$. Now, to define the orbital elements of Eq. (1), it is important to know that these elements are maintained in the neighborhood of the zero value by means of active control using stationkeeping maneuvers. The magnitude of these elements can, therefore, be assumed as follows:

$$|g_1|, |h_1|, |h_2| \leq O(m^2) \quad |e_1|, |e_2| \leq O(m^3) \quad |g_2| \leq O(m^4) \tag{12}$$

In this case, it is sufficient to expand Eqs. (8) and (9) up to quadratic terms in (h_1, h_2, e_1, e_2). Using the equation of the center,[18] we get

$$l_1 = l + 2(e_1 \sin l - e_2 \cos l) + (5/4)[(e_1^2 - e_2^2)\sin 2l - 2e_1 e_2 \cos 2l] \tag{13}$$

Substitutions of Eq. (13) in Eqs. (8) and (9) lead to

$$a/r = 1 + e_1 \cos l + e_2 \sin l + (e_1^2 - e_2^2)\cos 2l + 2e_1 e_2 \sin 2l$$

$$x = \cos l + (\cos 2l - 1)e_1 + e_2 \sin 2l - (1/8)(4h_2^2 + 9e_1^2 + 7e_2^2)\cos l + (1/4)(2h_1 h_2 - e_1 e_2)\sin l + (9/8)[(e_1^2 - e_2^2)\cos 3l + 2e_1 e_2 \sin 3l]$$

$$y = \sin l - (\cos 2l + 1)e_2 + e_1 \sin 2l - (1/8)(4h_1^2 + 7e_1^2 + 9e_2^2)\sin l + (1/4)(2h_1 h_2 - e_1 e_2)\cos l + (9/8)[(e_1^2 - e_2^2)\sin 3l - 2e_1 e_2 \cos 3l]$$

$$z = h_1 \sin l - h_2 \cos l + h_2 e_1 - h_1 e_2 + (h_1 e_1 - h_2 e_2)\sin 2l - (h_1 e_2 + h_2 e_1)\cos 2l \tag{14}$$

Equations (14) are now suitable for substitution in Eqs. (3-7). To simplify the resulting expressions, all fluctuations will be omitted from the coefficients of the quadratic terms of the orbital elements (h_1, h_2, e_1, e_2). This leads to percentage errors in $\Delta h_i/h_i$ and $\Delta e_i/e_i$ of amplitudes of $O(100\epsilon_m/\omega_i)$, where ω_i is the angular frequency of the omitted fluctuation. In the present development, these errors are considered to be small. Now, the substitution of Eqs. (14) in Eqs. (3-7) was made using the REDUCE program[3] of algebraic manipulation. The results are summarized in Appendix B of Ref. 14 for R_{m2}, R_{m3}, and R_{m4}. The sun's disturbing function R_{s2} has the same form as R_{m2} and can be obtained by replacing the subscript m by s. The oblateness disturbing function R_{obl} has the following simple form:

$$R_{obl} = \epsilon_{obl}[-2g_2 - \tfrac{1}{2}(h_1^2 + h_2^2) + \tfrac{1}{2}(e_1^2 + e_2^2) + e_1 \cos l + e_2 \sin l] \tag{15}$$

The next step in the development of the perturbation solution is to substitute the approximate moon ephemeris of Appendix A of Ref.14 in R_{m2}, R_{m3}, and R_{m4} of Appendix B and the sun ephemeris in R_{s2} (same form as R_{m2}). This is done as follows: The luni-solar ephemerides are used to its maximum order $[O(m^3)$ for the sun and $O(m^2)$ for the moon] in the computation of the long period fluctuations. For the short period oscillations (daily oscillations), the luni-solar ephemerides are used up to $O(m^2)$. In this case, the accuracy of the final solution is of $O(\epsilon_m m^2/\omega_e) \cong O(1.6 \times 10^{-7})$ rad. At synchronous altitude, this accuracy corresponds to about 7 m. Now, the quadratic part of the Hamiltonian that results from the substitutions, mentioned above, is given by

$$H_2 = \tfrac{1}{2}(\omega_{h_1} h_1^2 + \omega_{h_2} h_2^2 + \omega_{e_1} e_1^2 + \omega_{e_2} e_2^2) \tag{16}$$

where

$$\omega_{h_1} = -\epsilon_{obl} - 0.5129406\epsilon_s - [0.5087960 + 0.0256294(a_{syn}/a_m)^2]\epsilon_m = -3.102815E-04$$

$$\omega_{h_2} = -\epsilon_{obl} - 0.6316600\epsilon_s - [0.6265596 + 0.8222876(a_{syn}/a_m)^2]\epsilon_m = -3.288817E-04$$

$$\omega_{e_1} = \epsilon_{obl} + 0.8690320\epsilon_s + [0.8617438 + 1.25631(a_{syn}/a_m)^2]\epsilon_m = 3.646174E-04$$

$$\omega_{e_2} = \epsilon_{obl} + 0.2754324\epsilon_s + [0.2729048 - 0.5778206(a_{syn}/a_m)^2]\epsilon_m = 2.742586E-04$$

The periods P_h and P_e of the free oscillations of (h_1, h_2) and (e_1, e_2) are determined by

$$P_h = 2\pi/\sqrt{\omega_{h_1}\omega_{h_2}} = 1.966899E04 \text{ days} = 53.85076 \text{ years}$$
$$P_e = 2\pi/\sqrt{\omega_{e_1}\omega_{e_2}} = 1.986923E04 \text{ days} = 54.39899 \text{ years} \tag{17}$$

The linear parts of the disturbing functions R_{m2}, R_{m3}, R_{m4}, and R_{s2} can be put in the matrix form

$$\begin{bmatrix} R_{m2} \\ R_{s2} \\ R_{m3} \\ R_{m4} \end{bmatrix} = \begin{bmatrix} \overset{\cos}{R_{m2h1}} & \overset{\sin}{R_{m2h2}} & \overset{\cos}{R_{m2e1}} & \overset{\sin}{R_{m2e2}} \\ R_{s2h1} & R_{s2h2} & R_{s2e1} & R_{s2e2} \\ R_{m3h1} & R_{m3h2} & R_{m3e1} & R_{m3e2} \\ R_{m4h1} & R_{m4h2} & R_{m4e1} & R_{m4e2} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} \overset{\cos}{R_{m20}} \\ R_{s20} \\ R_{m30} \\ R_{m40} \end{bmatrix} \tag{18}$$

where the cos and sin over a column indicate that the elements of this column are trigonometric series of cos and sin form, respectively. For example, the function R_{m2h1} has the form

$$R_{m2h1} = \sum_j r_{m2h1}^{(j)} \cos\theta_j \tag{19}$$

where

$$\theta_j = \alpha_{1j}\bar{l} + \alpha_{2j} l_m + \alpha_{3j}\phi_m + \alpha_{4j}\eta_m + \alpha_{5j}\xi_m + \alpha_{6j} l_s + \alpha_{7j}\phi_s \tag{20}$$

The coefficients $r_{m2h1}^{(j)}$ have the same subscript as the function R_{m2h1} but it is defined by the lowercase letter r. The arguments θ_j are combination of the satellite mean longitude l and the moon and sun angles defined in Appendix A of Ref. 14. Tables 1-8 of Ref. 14 show the numerical values of the coefficients of the various functions of Eq. (18) as a function of $(\alpha_1, \alpha_2, \ldots, \alpha_7)$ defined by Eq. (20). Now, Eqs. (1-6), (15-18), and Tables 1-8 of Ref. 14 can be used to develop the required satellite ephemeris. This is the subject of the next two sections.

IV. Solution of (h_1, h_2, e_1, e_2)

In view of Eqs. (2), (5), (6), (16), and (18), the Hamiltonian part containing h_1 and h_2 is given by

$$H_h = \tfrac{1}{2}(\omega_{h_1} h_1^2 + \omega_{h_2} h_2^2) + F_{h_1} h_1 + F_{h_2} h_2 \tag{21}$$

where ω_{h_1} and ω_{h_2} are given by Eq. (16) and for $i=1,2$

$$F_{hi} = \epsilon_m [R_{m2hi} + (a_{\rm syn}/a_m) R_{m3hi} + (a_{\rm syn}/a_m)^2 R_{m4hi}] + \epsilon_s R_{s2hi} \tag{22}$$

In view of Tables 1-4 of Ref. 14, F_{hi} can be represented by the following trigonometric series

$$F_{h_1} = \sum_j f_{h_1}^{(j)} \cos\theta_j \qquad F_{h_2} = \sum_j f_{h_2}^{(j)} \sin\theta_j \tag{23}$$

where, for $i=1,2$

$$f_{hi}^{(j)} = \epsilon_m [r_{m2hi}^{(j)} + (a_{\rm syn}/a_m) r_{m3hi}^{(j)} + (a_{\rm syn}/a_m)^2 r_{m4hi}^{(j)}] + \epsilon_s r_{s2hi}^{(j)}$$

and θ_j's are given by Eq. (20).

Now, using Eqs. (1) and (21), the equations governing the motion of (h_1, h_2) are obtained in the form

$$\dot h_1 = -\omega_{h_2} h_2 - F_{h_2} \qquad \dot h_2 = \omega_{h_1} h_1 + F_{h_1} \tag{24}$$

These equations satisfy the solution

$$h_1 = a_h \sin\omega_h t + b_h \cos\omega_h t + h_{1e} + \sum_j c_{hj}\cos\theta_j$$
$$h_2 = \delta_h [b_h \sin\omega_h t - a_h \cos\omega_h t] + \sum_j s_{hj} \sin\theta_j \tag{25}$$

where

$$\delta_h = -\sqrt{\omega_{h_1}/\omega_{h_2}} = -0.9713105$$

$$\omega_h = \sqrt{\omega_{h_1}\omega_{h_2}} = 3.194462\text{E}-04 \text{ rad/day}$$

$$h_{1e} = -[0.2715685 + 0.3565043(a_{\rm syn}/a_m)^2](\epsilon_m/\omega_{h_1}) - 0.2738293(\epsilon_s/\omega_{h_1}) = 0.1323758$$

$$c_{hj} = \frac{\omega_j f_{h_2}^{(j)} + \omega_{h_2} f_{h_1}^{(j)}}{\omega_j^2 - \omega_h^2} \qquad s_{hj} = \frac{\omega_j f_{h_1}^{(j)} + \omega_{h_1} f_{h_2}^{(j)}}{\omega_j^2 - \omega_h^2} \tag{26}$$

$$\omega_j = \dot\theta_j = \alpha_{1j}\omega_e + \alpha_{2j}\omega_{m1} + \alpha_{3j}\omega_{m2} + \alpha_{4j}\omega_{m3} + \alpha_{5j}\omega_{m4} + \alpha_{6j}\omega_{s1} + \alpha_{7j}\omega_{s2} \tag{27}$$

$\omega_e = 6.300388$ and ω_{mi} $(i=1,2,3,4)$ and ω_{si} $(i=1,2)$ are given in Appendix A of Ref. 14.

The constants a_h and b_h are determined from the initial conditions h_{10} and h_{20} in the form

$$b_h = h_{10} - h_{1e} - \sum_j c_{hj}\cos\theta_j(t_0)$$
$$a_h = \delta_h^{-1}\Big[-h_{20} + \sum_j s_{hj}\sin\theta_j(t_0)\Big] \tag{28}$$

where $\theta_j(t_0)$ are the values of θ_j at the epoch t_0 as shown in Appendix A of Ref. 14.

Table 9 of Ref. 14 shows the coefficients c_{hj} and s_{hj} as function of $(\alpha_{1j}, \alpha_{2j}, \ldots, \alpha_{7j})$ defined by Eq. (20).

Now, the differential equations governing the motion of (e_1, e_2) have the same form as Eq. (24). In this case, however, there is an additional daily fluctuation due to the Earth's oblateness given by Eq. (15). The final solution is given by

$$e_1 = a_e \sin\omega_{\rm ecc} t + b_e \cos\omega_{\rm ecc} t + c_1 \cos l + \sum_j c_{ej}\cos\theta_j$$
$$e_2 = \delta_e [b_e \sin\omega_{\rm ecc} t - a_e \cos\omega_{\rm ecc} t] + s_1 \sin l + \sum_j s_{ej}\sin\theta_j \tag{29}$$

Using Eq. (16) we get

$$\delta_e = \sqrt{\omega_{e_1}/\omega_{e_2}} = 1.153025$$

$$\omega_{\rm ecc} = \sqrt{\omega_{e_1}\omega_{e_2}} = 3.162269\text{E}-04 \text{ rad/day}$$

$$c_1 = \frac{\omega_e + \omega_{e2}}{\omega_e^2 - \omega_{\rm ecc}^2}\epsilon_{\rm obl} = 3.716164\text{E}-05$$

$$s_1 = \frac{\omega_e + \omega_{e1}}{\omega_e^2 - \omega_{\rm ecc}^2}\epsilon_{\rm obl} = 3.716217\text{E}-05$$

The c_{ej} and s_{ej} are analogous to c_{hj} and s_{hj} of Eq. (26). The numerical values of these coefficients are given in Table 10 of Ref. 14 as function of $(\alpha_{1j}, \alpha_{2j}, \ldots, \alpha_{7j})$ of Eq. (20).

V. Solution of (g_1, g_2)

The Hamiltonian part that contains (g_1, g_2) can be obtained, using Eqs. (2-6), (15), and (18), in the form

$$H_g = \tfrac{1}{2}(\omega_{g_1} g_1^2 + \omega_{g_2} g_2^2) + F_{g_1} g_1 + F_{g_2} g_2 \tag{30}$$

where

$$\omega_{g_1} = -\epsilon_{\rm tess} G_2 \qquad \omega_{g_2} = 3\omega_e$$

$$F_{g_1} = -\epsilon_{\rm tess} G_1 + \epsilon_s \frac{\partial R_{s20}}{\partial l} + \epsilon_m \left[\frac{\partial R_{m20}}{\partial l} + \left(\frac{a_{\rm syn}}{a_m}\right)\frac{\partial R_{m30}}{\partial l} + \left(\frac{a_{\rm syn}}{a_m}\right)^2 \frac{\partial R_{m40}}{\partial l}\right] \tag{31}$$

$$F_{g2} = -2\epsilon_{\rm obl} - \epsilon_{\rm tess} G_3 + 4\epsilon_s R_{s20} + 2\epsilon_m \left[2R_{m20} + 3\left(\frac{a_{\rm syn}}{a_m}\right) R_{m30} + 4\left(\frac{a_{\rm syn}}{a_m}\right)^2 R_{m40}\right]$$

The Hamiltonian H_g has the same form as H_h of Eq. (21). In this case, however, F_{g1} is described by the sin series and F_{g2} by the cos series. Also, since F_{g1} is solely determined by diurnal fluctuations, g_1 and g_2 can be written in the form

$$g_1 = a_g \omega_{g2} f_1 + b_g f_2 - \omega_{g0}\omega_{g2} f_3 - \sum_{\alpha_{1j}\neq 0} s_{gj}\sin\theta_j - \sum_{\alpha_{1j}=0} \bar s_{gj}\sin\theta_j$$
$$g_2 = (\omega_{g0} + b_g \omega_{g1}) f_1 - a_g f_2 + g_{2e} + \sum_{\alpha_{1j}\neq 0} c_{gj}\cos\theta_j \tag{32}$$

where $\alpha_{1j} \neq 0$ indicate that the series is diurnal [see Eqs. (20) and (27)].

$$G_2 < 0: \quad f_1 = \frac{\sin(\omega g t)}{\omega_g} \qquad f_2 = \cos\omega_g t \qquad f_3 = \frac{[1 - \cos(\omega g t)]}{\omega_{g1}\omega_{g2}}$$

$$G_2 > 0: \quad f_1 = \frac{\sinh(\omega g t)}{\omega_g} \qquad f_2 = \cosh\omega_g t \qquad f_3 = \frac{[1 - \cosh(\omega g t)]}{\omega_{g1}\omega_{g2}}$$

$$G_2 = 0: \quad f_1 = t \qquad f_2 = 1 \qquad f_3 = \tfrac{1}{2}t^2$$

$$\omega_{g0} = -\epsilon_{\rm tess} G_1$$

$$\omega_g = \sqrt{|\omega_{g_1}\omega_{g_2}|} = 2.690623\mathrm{E}-03\sqrt{|G_2|}$$

$$g_{2e} = (1/3\omega_e)\{2\epsilon_{\rm obl} + \epsilon_{\rm tess}G_3 - 4\epsilon_s\bar{r}_{s20} - 2\epsilon_m[2\bar{r}_{m20} + 4(a_{\rm syn}/a_m)^2\bar{r}_{m40}]\}$$

$$\alpha_{1j}\neq 0: \quad s_{gj} = [f_{g2j} + 3\omega_e c_{gj}]/\omega_j$$

$$\alpha_{1j}\neq 0: \quad c_{gj} = -f_{gj}/\omega_j$$

$$\alpha_{1j} = 0: \quad \bar{s}_{gj} = s_{gj}[1 - \omega_{g1}\omega_{g2}/\omega_j^2]^{-1} \tag{33}$$

$\bar{r}_{s20}$, $\bar{r}_{m20}$, and $\bar{r}_{m40}$ are the constant terms obtained from Tables 1-4 of Ref. 14. Substitution with numerical values leads to

$$G_1 = 4\sin 2(\lambda_{\rm syn} - \lambda_{22}) - 0.1854368\sin(\lambda_{\rm syn} - \lambda_{31}) + 0.5734273\sin 3(\lambda_{\rm syn} - \lambda_{33})$$

$$G_2 = 8\cos 2(\lambda_{\rm syn} - \lambda_{22}) - 0.1854368\cos(\lambda_{\rm syn} - \lambda_{31}) + 1.720282\cos 3(\lambda_{\rm syn} - \lambda_{33})$$

$$G_3 = 12\cos 2(\lambda_{\rm syn} - \lambda_{22}) - 1.483494\cos(\lambda_{\rm syn} - \lambda_{31}) + 1.529139\cos 3(\lambda_{\rm syn} - \lambda_{33}) \tag{34}$$

$$g_{2e} = 1.876676\mathrm{E}-05 + 2.431700\mathrm{E}-07\cos 2(\lambda_{\rm syn} - \lambda_{22}) - 3.006176\mathrm{E}-08\cos(\lambda_{\rm syn} - \lambda_{31}) + 3.098673\mathrm{E}-08\cos - 3(\lambda_{\rm syn} - \lambda_{33})$$

In view of the definition of g_2 given by Eq. (1), there is a constant shift in the synchronous semimajor axis reminiscent of the constant shift in the moon's orbital semimajor axis produced by solar perturbations. This shift is given by

$$\delta a_{\rm syn} = 2a_{\rm syn}g_{2e} = 1.582573 + 2.050617\mathrm{E}-02\cos 2(\lambda_{\rm syn} - \lambda_{22}) - 2.535064\mathrm{E}-03\cos(\lambda_{\rm syn} - \lambda_{31}) + 2.613065\mathrm{E}-03\cos 3(\lambda_{\rm syn} - \lambda_{33})\ \text{km} \tag{35}$$

which is in good agreement with Ref. 5.

The coefficients s_{gj}, c_{gj} of Eq. (33) are given by Table 9 of Ref. 14 in terms of $\alpha_1, \alpha_2, ..., \alpha_7$ defined by Eqs. (20) and (27). In the computation of these coefficients, the ratio $(\omega_g/\omega_j)^2$ is ignored when $\alpha_{1j}\neq 0$. When $\alpha_{1j}=0$, however, this ratio could be significant for some station longitudes $\lambda_{\rm syn}$. To reflect this significant effect, this ratio is kept in the computation of $\bar{s}_{gj}$ when $\alpha_{1j}=0$. The coefficients s_{gj} are given in Table 9 of Ref. 14. To compute $\bar{s}_{gj}$, s_{gj} is simply multiplied by the adjustment factor $[1-(\omega_g/\omega_j)^2]^{-1}$. For station longitudes leading to $1-(\omega_g/\omega_j)^2=0$, one should replace $\bar{s}_{gj}\sin\theta_j$ by $\frac{1}{2}s_{gj}(\omega_j t)\cos\theta_j$ in Eq. (32).

VI. Conclusion

The perturbed motion of geosynchronous satellites has been developed in a form useful for rapid and precise determination of north-south and east-west stationkeeping strategies for geosynchronous satellites. The obtained solution can also be used in the assessment of the long-term orbital behavior of these satellites with a moderate degree of accuracy.

Acknowledgments

The author wishes to thank Prof. Roger Broucke of the University of Texas and Dr. Victor Slabinski of COMSAT for their contributions to this paper. Typing of the paper manuscript by Mrs. Sue Clarkson is greatly appreciated.

References

[1] Shrivastava, S.K., "Orbital Perturbations and Stationkeeping of Communication Satellites," *Journal of Spacecraft and Rockets*, Vol. 15, March 1978, pp. 67-78.

[2] Hearn, A.C., "A New REDUCE Model for Algebraic Simplification," *Proceedings of SYSMAC 76*, ACM, New York, 1976, pp. 46-52.

[3] Hearn, A.C., "REDUCE Manual," Department of Computer Science, University of Utah, Salt Lake City, Utah, March 1973.

[4] Cefola, P.J., "Equinoctial Orbit Elements—Application to Artificial Satellite Orbit," Paper 72-937 presented at the AIAA Astrodynamics Conference, Palo Alto, Calif., Sept. 11-12, 1972.

[5] Cefola, P.J. and Broucke, R., "On the Formulation of the Gravitational Potential in Terms of Equinoctial Variables," Paper 75-9 presented at the AIAA 13th Aerospace Sciences Meeting, Pasadena, Calif., Jan. 20-22, 1975.

[6] Brouwer, D. and Clemence, G., *Methods of Celestial Mechanics*, Academic Press, New York, 1961, pp. 562-573.

[7] Kamel, A., Ekman, D., and Tibbitts, R., "East-West Stationkeeping Requirements of Nearly Synchronous Satellites Due to Earth's Triaxiality and Luni-Solar Effects," *Celestial Mechanics*, Vol. 8, 1973, pp. 129-148.

[8] Kaula, W.M., *Theory of Satellite Geodesy*, Blaisdell Publishing Company, Waltham, Mass., 1966, pp. 49-56.

[9] Giacaglia, G.E., "Lunar Perturbations of Artificial Satellites of the Earth," *Celestial Mechanics*, Vol. 9, 1974, pp. 239-267.

[10] Kamel, A. and Tibbitts, R., "Some Useful Results on Initial Node Locations for Near-Equatorial Circular Satellite Orbits," *Celestial Mechanics*, Vol. 8, 1973, pp. 45-73.

[11] Kozai, Y., "A New Method to Compute Luni-Solar Perturbations in Satellite Motions," Smithsonian Astrophysical Observatory, Special Rept. 349, 1973.

[12] Musen, P., Bailier, P., and Upton, E., "Development of the Lunar and Solar Perturbations in the Motion of an Artificial Satellite," NASA TN-D-494, Jan. 1961.

[13] Richardson, D.L., "The Long-Period Motion of 24-Hour Satellites," AIAA Paper 76-828, San Diego, Calif., Aug. 1976.

[14] Kamel, A., "Synchronous Satellite Ephemeris Due to Earth's Triaxiality and Luni-Solar Effect," AIAA Paper 78-1441, Palo Alto, Calif. Aug. 1978.

[15] Kamel, A.A. and Wagner, C.A., "On the Orbital Eccentricity Control of Synchronous Satellites," Paper 81-128, presented at the AAS/AIAA Astrodynamics Specialist Conference, Lake Tahoe, Nev., Aug. 3-5, 1981.

[16] Gaposchkin, E.M., "1972 Smithsonian Institute Standard Earth," *Journal of Geophysical Research*, Vol. 79, 1974, pp. 5377-5411.

[17] Escobal, P.R., *Methods of Astrodynamics*, John Wiley and Sons, Inc., New York, 1968, pp. 318-320.

[18] Smart, W.M., *Celestial Mechanics*, John Wiley and Sons, Inc., New York, 1953, p. 38.

[19] Brown, E., *An Introductory Treatise on the Lunar Theory*, Cambridge University Press, New York, 1896, p. 82.